IMONST 2 2020
SENIOR CATEGORY

Instructions

1. The IMONST 2 paper contains six (6) essay/proof-type problems. Each problem is worth 7 points each. Integer partial scores are possible.
2. There are different papers for Primary, Junior, and Senior categories. Please make sure you have the paper for the right category.
3. The time allowed for the test is 5 hours, from 10.00 am to 3.00 pm.
4. You need:
   (a) An internet connection to download the question paper and submit your solutions.
   (b) A4 papers (blank, not lined) for scratch papers and final scripts.
   (c) Pens, pencils, erasers, geometric tools (compass, ruler).
   (d) IMPORTANT: A scanner to scan your scripts. If you don't have a scanner, please download a scanner app on your phone – we recommend Adobe Scan.
5. You are permitted to use a calculator, but this not essential.
6. Please use your handwriting to write the solutions. Do not type or use any word processing/typesetting software to produce your script.
7. Write your **full name, category, and school/institution** at the **top of the first page**.
8. This is an open book test. You may refer to any internet articles or printed material.
9. Work on the solutions on your own. Any form of outside help, such as discussions, communication, or assistance from another person, is not allowed.
10. Please follow the instructions on the contest page to submit your answer scans.

IMONST 2 2020 (Senior)

Problem 1. Given a trapezium with two parallel sides of lengths m and n, where m and n are integers. Prove that it is possible to divide the trapezium into several congruent triangles.

Note: Two triangles are congruent if they have the same shape and the same size.

Diberi suatu trapezium dengan dua sisi selari yang panjangnya m dan n, dengan m dan n integer. Buktikan bahawa kita boleh membahagikan trapezium tersebut kepada beberapa segitiga yang kongruen.

Nota: Dua segitiga adalah kongruen jika kedua-duanya mempunyai bentuk dan saiz yang sama.

Problem 2. Prove that

Buktikan bahawa

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots + \frac{1}{2019} - \frac{1}{2020} = \frac{1}{1011} + \frac{1}{1012} + \frac{1}{1013} + \cdots + \frac{1}{2020}.$$

Problem 3. Find all possible integer values of n such that $12n^2 + 12n + 11$ is a 4-digit number with all 4 digits equal.

Cari semua nilai integer yang mungkin bagi n sehinggakan $12n^2 + 12n + 11$ adalah suatu nombor 4-digit dengan 4 digit yang sama.

Problem 4. Given four circles Γ_1, Γ_2, Γ_3, and Γ_4. Circles Γ_1 and Γ_2 are externally tangent at point A. Circles Γ_2 and Γ_3 are externally tangent at point B. Circles Γ_3 and Γ_4 are externally tangent at point C. Circles Γ_4 and Γ_1 are externally tangent at point D.

Prove that quadrilateral $ABCD$ is cyclic.

Diberi empat bulatan Γ_1, Γ_2, Γ_3, dan Γ_4. Bulatan Γ_1 dan Γ_2 adalah tangen secara luaran pada titik A. Bulatan Γ_2 dan Γ_3 adalah tangen secara luaran pada titik B. Bulatan Γ_3 dan Γ_4 adalah tangen secara luaran pada titik C. Bulatan Γ_4 dan Γ_1 adalah tangen secara luaran pada titik D.

Buktikan bahawa $ABCD$ adalah suatu sisiempat kitaran.

Problem 5. Let p and q be real numbers such that the quadratic equation $x^2 + px + q = 0$ has two distinct real solutions x_1 and x_2. Suppose that $|x_1 - x_2| = 1$ and $|p - q| = 1$. Prove that all of p, q, x_1, and x_2 are integers.

Andaikan p dan q nombor nyata sehinggakan persamaan kuadratik $x^2 + px + q = 0$ mempunyai dua penyelesaian nyata x_1 dan x_2 yang berbeza. Andaikan bahawa $|x_1 - x_2| = 1$ dan $|p - q| = 1$. Buktikan bahawa kesemua nombor p, q, x_1, dan x_2 adalah integer.

■ **CONTINUES ON NEXT PAGE** ■

Problem 6. Consider the following one-person game: A player starts with score 0 and writes the number 20 on an empty whiteboard. At each step, she may erase any one integer (call it a) and writes two positive integers (call them b and c) such that $b + c = a$. The player then adds $b \times c$ to her score. She repeats the step several times until she ends up with all 1s on the whiteboard. Then the game is over, and the final score is calculated.

Example: At the first step, a player erases 20 and writes 14 and 6, and gets a score of $14 \times 6 = 84$. In the next step, she erases 14, writes 9 and 5, and adds $9 \times 5 = 45$ to her score. Her score is now $84 + 45 = 129$. In the next step, she may erase any of the remaining numbers on the whiteboard: 5, 6, or 9. She continues until the game is over.

Alya and Bob play the game separately. Alya manages to get the highest possible final score. Bob, however, manages to get the lowest possible final score. What is the difference between Alya's and Bob's final scores?

Pertimbangkan permainan berikut yang melibatkan seorang pemain: Pemain tersebut bermula dengan markah 0 dan menulis nombor 20 di suatu papan putih yang kosong. Pada setiap langkah, dia boleh memadamkan mana-mana satu nombor (dipanggil a) dan menulis dua integer positif (dipanggil b dan c) dengan $b+c = a$. Pemain tersebut kemudian menambahkan $b \times c$ kepada markahnya. Dia mengulang langkah ini beberapa kali sehingga dia mendapat semua nombor 1 pada papan putih tersebut. Pada waktu ini, permainan tersebut berakhir dan markah akhirnya dikira.

Contoh: Pada langkah pertama, seorang pemain boleh memadamkan 20 dan menulis 14 dan 6, dan mendapat markah $14 \times 6 = 84$. Pada langkah seterusnya, dia memadam 14, menulis 9 dan 5, dan menambahkan $9 \times 5 = 45$ kepada markahnya. Markahnya sekarang ialah $84+45 = 129$. Pada langkah seterusnya, dia boleh memadamkan mana-mana nombor yang terdapat di papan tersebut: 5, 6, atau 9. Dia meneruskan langkah-langkah ini sehingga permainan itu berakhir.

Alya dan Bob bermain permainan tersebut secara berasingan. Alya berjaya mendapat markah akhir paling tinggi yang boleh dicapai. Bob pula berjaya mendapat markah akhir paling rendah yang boleh dicapai. Apakah beza antara markah akhir Alya dan Bob?

■ **END OF PAPER** ■